Let's Play MATH

How Families Can Learn Math Together—and Enjoy It

DENISE GASKINS

FOREWORD BY KEITH DEVLIN

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The best way to learn mathematics is to follow the road
which the human race originally followed:
Do things,
make things,
notice things,
arrange things,
and only then—reason about things.

Above all, do not try to hurry.
Mathematics, as you can see, does not advance rapidly.

The important thing is to be sure
that you know what you are talking about:
to have a clear picture in your mind.
Keep turning things over in your mind
until you have a vivid realization of each idea.

When we find ourselves unable to reason
(as one often does when presented with, say, a problem in algebra)
it is because our imagination is not touched.
One can begin to reason only when a clear picture
has been formed in the imagination.

Bad teaching is teaching which presents an endless procession
of meaningless signs, words, and rules,
and fails to arouse the imagination.

— W. W. Sawyer

Contents

Foreword

Authors and publishers of new mathematics and math education books frequently ask me to write a foreword or a cover endorsement. Denise Gaskins's *Let's Play Math: How Families Can Learn Math Together and Enjoy It* is not one of those cases. I volunteered to write a foreword. I did so because I wanted to help in any way I could to get this book into the hands of as many parents and math educators as possible, particularly the hands of the large number of homeschooling parents in the USA—many of whom purchase some of my books or listen regularly to my Math Guy segments on NPR, and some of whom email me and attend public talks I give around the country.

For, in the publishing world, the odds are stacked against Ms. Gaskins. She is not a professor at a major university, nor indeed any institute of higher education. Nor is she an "award winning career teacher." On top of which, she does not have a major publisher behind her. I could have put a "yet" in that last sentence, but I'm not at all sure her future success with this book will play out that way, though in terms of sheer quality it could if she wanted.

"So who is she?" you may ask. On the Amazon page for the first edition of this book, which she self-published under the banner Tabletop Academy Press, she describes herself this way:

> *Denise Gaskins is a veteran homeschooling mother of five who has taught or tutored at every level from preschool to undergraduate physics. She loves math, and she delights in sharing that love with young people.*

Here is how she summarized her mathematics education activities when I asked her for a bit more detail:

1978-1984: assorted jobs including volunteer tutoring, physics T.A., and a one-semester stint as a 6th-grade teacher in a private school

1982 (I think): B.S. in physics & science writing, Purdue University
1984-PRESENT: homeschooling mother of five
1992: began writing sporadically about math education
1995-2014: led math circles or math classes for local homeschoolers
1998: published my first booklets to accompany math workshops for homeschool groups
1998-2001: published bimonthly *Mathematical Adventures* newsletter for homeschoolers
2006: started Let's Play Math blog, originally aimed at homeschooling parents but the audience has widened over the years
2009: started Math Teachers at Play blog carnival to support creative math education in families and classrooms
2012: published *Let's Play Math* ebook first edition
2015: contributing author to Sue VanHattum's *Playing with Math: Stories from Math Circles, Homeschoolers, and Passionate Teachers*
2015: published two *Math You Can Play* books of number games (*Counting & Number Bonds* and *Addition & Subtraction*) in ebook and paperback

I deliberately left in the caveat in her 1982 entry as I find it particularly revealing. The physics major and the science writing, at one of the nation's top engineering universities, explain a lot about her success; the year is irrelevant, implying (at least to me) that she is not particularly interested in the university credential. Credentials play an important role in society, but they definitely get in the way of good education.

Denise and I have never met, but having followed her blog Let's Play Math for over eight years, I feel I know her. I first encountered her back in 2008, when I wrote a series of articles in my online Devlin's Angle column for the Mathematical Association of America, asking teachers to stop teaching multiplication as repeated addition.

Why did I suggest that? Because it isn't. See my MAA posts for June, July-August, and September of 2008 for explanations of why it isn't and why it is harmful to teach it as such, and then January 2011 for a brief summary of what multiplication is.†

† *https://www.maa.org/external_archive/devlin/devlin_06_08.html*
https://www.maa.org/external_archive/devlin/devlin_0708_08.html
https://www.maa.org/external_archive/devlin/devlin_09_08.html
https://www.maa.org/external_archive/devlin/devlin_01_11.html

Those articles were not opinion pieces; I was simply reporting what has long been known about mathematics and math education. Yet many people reading them assumed they were, and argued vehemently, and often passionately, that I was wrong. But arguing based on your *existing beliefs* is the worst thing to do in education. Learning is about looking at the evidence, reflecting on it, cross-checking to ensure veracity of reporting, and then adjusting your knowledge and beliefs accordingly. If you are not discovering that you were wrong, or that you did not properly understand something, then *you are not learning.* Period.

Enter Denise into the fray. (It was actually more like a firestorm at the time.) Her July 1, 2008 post on Let's Play Math, which she wrote right after she read the first of my multiplication-is-not-repeated-addition posts, was one of the best illustrations of how the mathematics learning process *should* progress I have ever seen. Check it out, paying particular attention to her *process.* She came back to the topic in two subsequent posts.[†]

In the months and years that followed, we had a (small) number of email exchanges, and followed each other's writings. She read me because I had devoted much of my life to mathematics, and *how to express it* so as to make some of its deeper complexities accessible to a wider audience; I read her because she had devoted much of her life to *how to teach it* to younger people so that they can learn it. Teaching is not instruction, though many fail to see the distinction. Teaching is creating the circumstances in which a person can learn. In the education domain, I am primarily an instructor; Denise is a teacher. We learned from each other, bringing different experiences and different perspectives.

Which brings me back to my opening comments. We are used to assigning labels such as "mathematician", "teacher", "writer", "doctor", "accountant", "journalist", etc., based on established credentials. This gives us a way to quickly judge how we should approach such individuals and whether to put our faith in what they say.

But in today's digitally-connected world, there is another path to achieving "professional" status. Instead of convincing a small number of

† *http://denisegaskins.com/2008/07/01/if-it-aint-repeated-addition/*
http://denisegaskins.com/2008/07/28/whats-wrong-with-repeated-addition/
http://denisegaskins.com/2012/07/16/pufm-1-5-multiplication-part-1/

people (examiners, editors, etc.) that we merit such approval, with each of those gatekeepers having undergone the same vetting process, now anyone can set themselves up as whatever they choose, put out or promote their work on the Internet, and then let the so-called "Wisdom of the Crowd" make the call.

It's a process fraught with dangers (so was the old system), and open to abuse and manipulation (so was the old system). But as Wikipedia showed, when it works it can be every bit as good as, if not better than, the old system.

Since anyone can play, this alternative approach is undoubtedly much more democratic than the older, establishment framework. The problem—and it is a big one—is that in the ocean of activity that is the Internet, it is hard for a truly talented individual to get their work noticed.

If it were not for the chance occurrence that Bill Gates stumbled across Salman Khan's online math instruction videos when he was trying to help his son with his math homework, Khan Academy would likely still be one of many largely unknown websites offering math tutorial videos. Were it not for Dan Meyer having been invited to give a TEDtalk that went viral, he would likely be to this day one of many blogging math teachers. And we can all think of other examples.

It's not that Sal Khan and Dan Meyer were not doing something of value. Rather, it required a stroke of luck to bring them to the attention of someone who could propel them far enough for their own talent to do the rest.

Denise's Let's Play Math blog, which is approaching 1,000 posts at the time of my writing this foreword (early 2016), gets around 40,000 page views per month (about 25,000 visitors), with about 1,300 blog feed subscribers. That's a successful blog. But it's nothing like where it should be in terms of the interest and quality of the posts.

Well, I'm not Bill Gates, nor do I control who gets invited to give TEDtalks. But insofar as I have some degree of name recognition in the math world, I'd like to use it to try to bring Denise Gaskins's work to a wider audience. That's why I offered to write this foreword and to promote her book in my various writings. It may not be enough; but I want to give it a try. [Note to other authors. Pitching me is not likely to work. Denise and I have been exchanging emails since 2008. She sent me a copy of her manuscript only after I requested it, which I did after she emailed me asking my permission to include a short passage from Devlin's Angle.]

At the time I am writing these words, the first edition of this book sits at position 315,714 in the Amazon ranking of paid Kindle books. Based on its quality, it should be much higher.

On the other hand, the first edition is (again at the time of writing) in the Top 100 in the Education & Teaching/Teacher Resources/Parent Participation category.

Take note of that categorization. Largely through her blog, Denise is known in the homeschooling community (though not exclusively so). And that is no small community. According to the US Department of Education's National Center for Education Statistics (NCES) 2013 report, just under 2 million students were being homeschooled in the US at the time, roughly 3.5% of the school-age population. That figure has surely grown since then.†

Parents homeschool for a variety of reasons, and do so with a wide range of abilities, doubtless with a wide range of success. For many, maybe most, mathematics presents a particularly difficult challenge. Dramatic changes in society and the workplace resulting from new computational technologies have rendered irrelevant much of the math it was important for my generation to learn, while at the same time making other math skills now critically important. I'm not referring just to "math content" here. The way we think about mathematics—the way we approach it—has changed. The emphasis used to be—correctly—on mastery of a set of procedures. Today, when we have access to all the resources on the Internet and have cheap devices in our pockets that can carry out those procedures faster and much more accurately than the human brain, the critical math abilities are conceptually sound mathematical thinking and creative problem solving, making use of the technologies to carry out the procedural parts. (Arguably good collaboration and communication skills are equally important, but they are of a different nature.)

Unfortunately, when it comes to mathematics, many homeschooling parents have little recourse other than fall back on how they themselves learned in school (or all too often, *failed* to learn in school). Even if they realize that they owe their children more, they don't know where to find

† *http://nces.ed.gov/pubsearch/pubsinfo.asp?pubid=2013316*

it or how to evaluate what they find. Show me a math resource you pick on the Web at random and I'll likely be able to point out a whole host of damaging errors of different kinds.

Denise knows what is involved in being a homeschooling parent. As a result, I would hope that other homeschoolers will take note of what she says. What is significant in her case, and is definitely not the case for many of the homeschooling parents I have met and interacted with over the years, is that she understands the mathematics and what is involved in teaching it.

Does she know it all? No. Neither do I. Is she always right? No. Neither am I. But neither of those matter. Mathematics is not about knowing or being right. It is about wanting to know and wanting to be right. It is about how we think about things, how we react to discovering we are wrong, and how we learn new things —new concepts, new facts, new procedures. It is about approaching everything with an open mind. It is about recognizing that doing math means constantly feeling we are about to fall off the cliff of comprehension, but approaching it in a want-to-win, playful way that lets us enjoy that "fear."

Let's Play Math: How Families Can Learn Math Together and Enjoy It is a special book. Its very title captures what I think is its most important feature: the book is a rich resource of ideas and activities for parents to explore and work through with their children. Written by a great writer who has been doing exactly that for many years.

"Math is not just rules and rote memory," Denise says. "Math is like ice cream, with more flavors than you can imagine. And if all you ever do is textbook math, that's like eating broccoli-flavored ice cream."

Enjoy (with your children) the fayre that Denise Gaskins serves up. After all, how often do you get an opportunity to feast—every day—on a family dessert that is highly nutritious and will help set your kids up for life?

—Keith Devlin
Palo Alto, CA
January 30, 2016

Preface to the Paperback Edition

As a young parent and newbie homeschooler, I tried to fit my children's education into the only model of school I knew. Textbooks, daily schedules, and slogging through one tedious workbook after another made education seem boring, when it ought to be a lifelong adventure. As my children grew, I noticed how much learning happened outside of "school" time, through library books or life experiences. We moved to a more relaxed, eclectic mentoring style. We discovered that even a math textbook can be fun when used as a source of puzzles.

On a daily basis, homeschoolers, tutors, and parents simply trying to help with homework experience the truth of the adage "The teacher learns more than the students." I've been learning more than my children for three decades now, and helping other parents learn more than their children for almost that long.

My math books began as handouts for my workshops and conference talks, folded and stapled by hand. When they grew too big for the stapler, I published them as simple, comb-bound paperbacks in the late-1990s. After those went out of print, I started my Let's Play Math blog to provide extra resources for my workshop participants. The old books sprouted as blog posts, fertilized by new tips, updated examples, and hands-on activities.

Through my blog, I discovered a wider audience. All parents, whatever their school (or unschool) affiliation, naturally want their children to enjoy learning. They are hungry for creative, playful ways to approach math. To my surprise, classroom teachers also were interested in what a homeschool mom had to share. We all face the same struggle: to explain abstract concepts in a way that young minds can grasp.

Meanwhile, four of my children grew up and graduated. My youngest is now in high school. After so many years of parenting, I'm still learning and thinking to myself, "I've got to share this!" So I mixed the fruits of my blog—revised games, creative projects, fresh insights—back into this *Let's Play Math* book and the *Math You Can Play* series. I've fixed all the typos I could find, deleted obsolete references, and served it all up with a tasty buffet of math books and Internet resources.

I'd Love to Hear from You

I hope my books help to make math your children's favorite subject. If you have any questions, please drop me a note.

—Denise Gaskins
Blue Mound, IL
November17, 2016
LetsPlayMath@gmail.com

Our own experience with
introducing advanced math to little kids
tells us that it can be difficult.

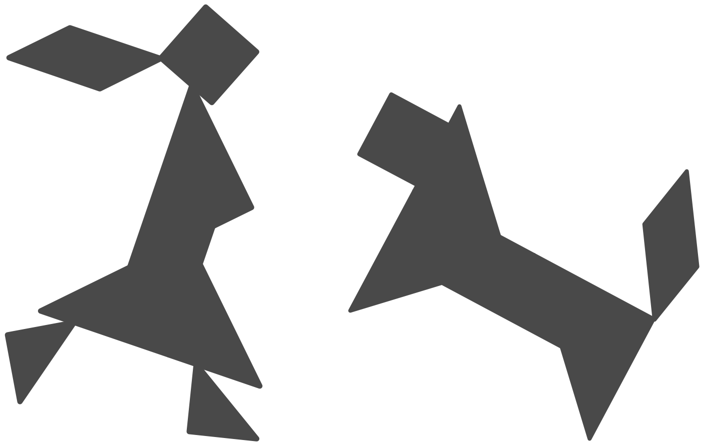
Surprisingly, the difficulty is not
in getting the kids
to understand the concepts.

Instead, it's the difficulty
in getting the non-mathematician parents
to believe that math can be fun
and to see it all around us.
After years and years of traditional math learning,
many parents find it hard to think of math
as something other than numbers.

Sure, math does deal with numbers.
But limiting mathematics to numbers
is like limiting parenting to changing diapers.

—Yelena McManaman

Section I



How to Understand Math

It is in fact nothing short of a miracle that the modern methods of instruction have not entirely strangled the holy curiosity of inquiry; for this delicate little plant, aside from stimulation, stands mainly in need of freedom.

—Albert Einstein

Introduction

A cup of coffee, a slice of pecan pie, and a robust discussion of educational philosophy—when I was a novice homeschooler, our local moms' night out provided mentoring and kept me sane. Years passed. Children grew. Many of the kids we worried over then are now raising children of their own. Though I can't remember growing older, I look in the mirror and find a gray-haired veteran.

I'd love to sit down with you for an afternoon's chat or an evening at the coffee shop, but our "night out" will have to be virtual. So I'll sip at my cup while I write. Perhaps you can nibble a bit of pie as you read. And together let's ponder the problem of learning math.

Our childhood struggles with schoolwork left many of us wary of mathematics. We learned to manipulate numbers and recite basic facts and formulas, but we never saw how or why it all fit together. We stumbled from one class to the next, packing ever more information into our strained memory, until the whole structure threatened to collapse. Eventually we crashed in a blaze of confusion, some of us in high school algebra, others in college calculus. If this is your experience, you may be wondering how you can possibly help your children learn math. Don't worry, you can! I'll show you that doing math together is easier than you think, and an awful lot of fun for both you and your children.

Before plunging in, let's take a moment to think about education.

Everyone has a philosophy of education, though they may not have thought it through. We've all been taught. There were parts of our schooling we liked and other parts we'd like to have changed. Over the years—in books, on websites and parenting forums, and in personal discussions—I've heard a range of opinions about how children learn math. Take a look at the list of statements I've collected, and think about the education you want for your children.

Which of the following points would you say are true? Which are math myths?

- Mathematics means the rules for working with numbers, shapes, and algebraic symbols.
- Math is in the genes. Some people have a "math mind," but most of us don't.
- Math is logical and rigid, not creative or artistic.
- Math is timeless and objective. It's the same for everyone.
- In mathematics, answers are either right or wrong. The right answer is never a matter of opinion.
- To do well at math, you need a good memory.
- Learning mathematics is like climbing a ladder. You have to master the basics before you can reach the higher rungs.
- Children need a textbook or workbook to learn math.
- Looking at someone else's answer is cheating.
- Students should show all the steps of their work. Shortcuts will lead to mistakes.
- Children shouldn't count on their fingers.
- Children need to memorize the times tables. They should drill the math facts until they can answer flashcard-fast.

These statements sum up the way many adults remember school mathematics. Yet they are all math myths. Not one of these statements is indisputably true.

Could these myths be the reason why so many children learn to hate math? Or why so many parents feel inadequate to help their kids?

"There is a huge elephant standing in most math classrooms," says Stanford University math education professor Jo Boaler. "It is the idea that only some students can do well in math. Students believe it, parents believe and teachers believe it. The myth that math is a gift that some students have and some do not, is one of the most damaging ideas that pervades education in the US and that stands in the way of students' math achievement."

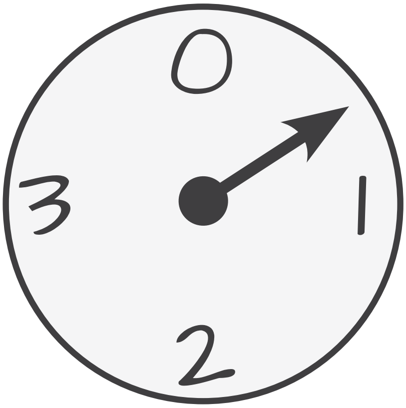
Unfortunately, math myths don't just stand there peacefully. Like a wild animal caught in a small room, they stamp and trample and wreak havoc on a child's confidence. Before we can help our children learn math, we need to chase these myths away.

Mathematics is much more than a set of rules. Contrary to popular perception, it can be very artistic. Unlike traditional school work, real math poses intriguing questions, making us want to explore its patterns and puzzles. As we play alongside our children, we can share the satisfaction of discovering why things work.

Yes, there are advanced topics that can be hard to understand. Some math problems are fiendishly difficult. Yet the basic principles of math—even at the high school level—grow from common sense. Learning them should feel natural.

Math is not the same for everyone, because what you see depends on your point of view. There are also different kinds of math. For example, can you draw a pair of parallel lines? In some versions of geometry, parallel lines do not exist. Does 2 + 2 = 4? Not always: in modular arithmetic 2 + 2 = 0 mod 4, while in base three 2 + 2 = 11. These topics can be interesting to explore with your children. Finding new ways to look at familiar ideas is part of the joy of learning math as a family.

A few of the myths in my list may have a semblance of truth, but their cumulative effect is to limit our children's understanding and ability to appreciate math. Many later math topics do build on earlier ones, but learning math is more like taking a meandering nature walk than like climbing a ladder with one rung above another. Preschool children are capable of exploring topics like fractals or infinity, while elementary students can begin learning algebra. It's fun to play with advanced ideas. Such



Modular arithmetic works like a one-handed clock. In mod 4, moving 2 + 2 brings you back around to zero. And 2 + 3 isn't five, because that number doesn't exist in mod 4. Instead, you keep moving around the clock, so 2 + 3 = 1. How many other mod 4 math facts can you find?

adventures offer a broad perspective that supports their knowledge of the more standard arithmetic topics.

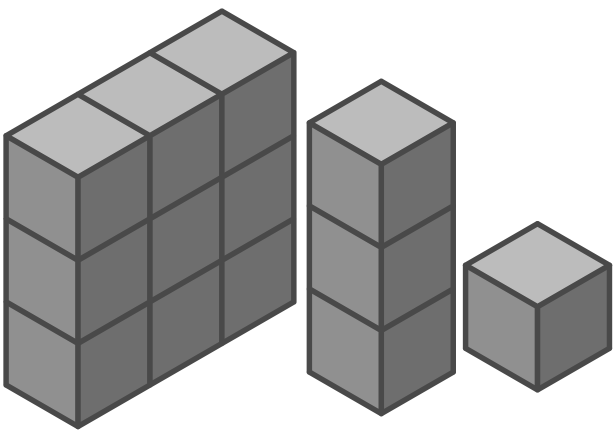
Even with the basics of *arithmetic* (number calculations, the traditional focus of elementary school math), the ladder analogy hurts more than it helps. Young children need freedom to wander from one topic to another as interest and opportunity lead them. They can ponder the concepts of multiplication and fractions long before they have finished mastering addition and subtraction.

Nor is a textbook necessary, at least during the early-elementary years. Most young children have a natural interest in mathematical ideas as part of their ongoing mission to understand and control their world. They find numbers fascinating, especially big numbers like hundreds or thousands. They enjoy drawing circles and triangles. They delight in scooping up volumes in the sandbox or bathtub. They can count out forks and knives for the table, matching sets of silverware with the resident set of people. They know how to split up the last bit of birthday cake and make sure they get their fair share, even if they have to cut halves or thirds.

Homeschoolers are not immune to math myths. When I was a child, all teachers exhorted us to "Show your work." I've seen many homeschooling discussion forum posts asking how parents can convince kids to write out the steps of their answers. Yet our teachers were not really interested in our childish pencil-scratchings; what they wanted was a window into how we were thinking. As parents, we have an advantage that classroom teachers can only dream of—namely, the time to sit and talk with each of our children. I can ask my daughter, "How did you figure it out?" In the course of conversation, she will demonstrate how much she knows. I often find myself learning something from the discussion, too, since she almost never thinks a problem through in the same way I would have done. I'd hate to trade this opportunity for a notebook page full of written-out steps.

Or consider the idea that looking at someone else's answer is cheating. While that is undoubtedly true during a test, such pressure should be rare. Wise parents and teachers know that children need to hear many different ways to approach a problem. They need to compare their solutions with others. When students are stumped on a math exercise, one of the best ways to learn is to look up the answer and work backward.

The last statement in my list of math myths—that children should memorize their times tables and practice until they can answer flashcard-fast—is the most controversial. With our modern culture's infatuation with test scores, many people will argue, "That's not a myth!" Frantic parents scour the Internet, desperate for tricks that will help their kids learn the math facts. But for many children, this emphasis on memory work does more harm than good. While understanding the meaning of multiplication is vital, instant recall is like icing on a brownie: tasty, but unnecessary. When we stress memorization, we risk short-circuiting our child's learning process. Once kids "know" an answer, they no longer



In base 3, the place value columns are multiples of three. Instead of writing numbers in terms of ones, tens, and hundreds, we count by ones, threes, and nines. Thus 2 + 2 = 11, because "11" means "one three and one single block." How many other base three facts can you find?

bother to think about it. It's better for them to spend more time in the "thinking about it" stage, where they can build a logical foundation for mastery not merely of the math facts but of many future topics as well.

Let's Play Math will show you how to build on your children's natural attraction to mathematical ideas. Even if you don't know much about mathematics—even if you had a terrible childhood experience of school math, even if the idea of fractions or long division makes you feel like throwing up—you can help your children master math. There's always time to make a fresh start. We adults can learn right along with our kids.

So turn the page, and let's explore what it means to understand math as a game. Along the way, you will discover practical ways to weave informal, playful math into your everyday lives, develop problem-solving skills, and bring math alive by exploring history. You'll find out how to deal with many of the common struggles children face and meet a wealth of resources to help you on your educational journey. Most of all, you will see how people of all ages can enjoy learning math together.

Links and Resources

All of the books and websites I mention are listed in the Resources and References section at the end of this book (beginning on page 211), or you can check out the resource pages on my blog:
http://denisegaskins.com/living-math-books
http://denisegaskins.com/internet-math-resources

The website links in this book were checked in December 2015, but the Internet is volatile. If a website disappears, you can run a browser search for the author's name or article title. Or try entering the web address at the Internet Archive Wayback Machine.
http://archive.org/web/web.php

1

If a child is to keep alive his inborn sense of wonder, he needs the companionship of at least one adult who can share it, rediscovering with him the joy, excitement, and mystery of the world we live in.

—Rachel Carson

The "Aha!" Factor

I sat on the bed, surrounded by stacks of notes, bills, and other papers. Our six-year-old bounced onto the other side.

"Mom, can we do some math?" she asked.

We delay academics, so I had not planned to do schoolwork with her at all. I suppose she was jealous of what she saw as her older siblings' Mommy-time.

I started to say I was busy, but stopped myself in mid-grumble. *You do that too often,* I scolded myself. I forced a smile.

"OK," I said. "Let's see what we can find."

I put the most important papers to the side. We counted the items that remained, then took some away and counted again.

Then I leaned over and whispered in her ear. "Guess what? We made cookies, just for you and me. And we're not going to share them with the big kids."

Her eyes grew wide. "Really?"

I told her we had six (imaginary) cookies. I piled up two notebooks, a used envelope, a sheet of scratch paper, a computer printout, and my pen. She divided the "cookies" between us. We giggled as we pretended to eat. Then she picked out seven new cookies for me to divide. I made exaggerated motions of cutting the electric bill in half. We counted things one by one and in pairs, paying attention to which numbers came out even and which numbers made us cut up the last cookie.

After ten minutes she went away happy, and I returned to my work. Subtraction, division, even and odd numbers, fractions, … in that short time, we had touched on more math than we might have found in a week's worth of workbook pages.

For young children, mathematical concepts are part of life's daily adventure. Their minds grapple with understanding abstract ideas such as *threeness:* the intangible yet real link between three blocks and three fingers and three raisins on a plate. But after a few sessions of "3 + 1 = 4, 3 + 2 = 5, 3 + 3 …" they begin to whine. Older children recoil from long division. High school students face torture like this: "The product of an integer and the next greater integer is 20 less than the square of the greater integer." Math becomes a tedious chore to put off as long as possible or to finish with slapdash speed.

Mathematics ought to be a game of discovery. It should give children the same *Eureka!* thrill that sent Archimedes running through town in his birthday suit. I call this the "Aha!" factor, the delight of solving a challenging puzzle. I aim for this "Aha!" factor when I bring home a brainteaser book from the library or a new game from the store.

The Problem with Traditional School Math

Why, as they grow up, do so many children learn to hate math? Why does the idea of math homework make so many parents feel like crying?

American mathematician Hassler Whitney once said that it is "no wonder you hate math. You never had a chance to see or do real math, which is easy and fun."

Easy? Yes, that's what he said. Of course, some parts of math can be difficult to master. Some math problems are extremely challenging. But compared to traditional school math, which requires us to memorize and recall arbitrary rules for the manipulation of abstract quantities, real mathematics is more like common sense—which makes it feel more natural.

As British mathematician, educator, and author W. W. Sawyer explained, "A widespread fallacy about teaching is the idea that remembering is easy and understanding difficult. John is a bright boy, we will teach him what the subject really means; Henry is dull, he will just have to learn things by heart. Now exactly the opposite is true: to remember things which you do not understand is extremely difficult."

Real mathematics is intriguing and full of wonder—an exploration of patterns and mysterious connections. It rewards us with the joy of the "Aha!" feeling. These characteristics make it easy to stick with real math, even when a particular concept or problem presents a difficult challenge. Workbook math, on the other hand, is several pages of long division by hand followed by a rousing chorus of the fraction song: "Ours is not to reason why, just invert and multiply."

Real math is the surprising fact that the odd numbers add up to perfect squares (1, 1 + 3, 1 + 3 + 5, etc.) and the satisfaction of seeing why it must be so. Did your algebra teacher ever explain to you that a *square number* is literally a number that can be arranged to make a square? Try it for yourself:

- ♦ Gather a bunch of pennies, or any small items that will not roll away when you set them out in rows. Place one of them in front of you on the table. Imagine drawing a frame around it: one penny makes a (very small) square. One row, with one item in each row.
- ♦ Now, put out three more pennies. How will you add them to the first one to form a new, bigger square? Arrange them in a small L-shape around the original penny to make two rows with two pennies in each row.
- ♦ Set out five more pennies. Without moving the current four, how can you place these five to form the next square? Three rows of three.
- ♦ Then how many will you have to add to make four rows of four?

Each new set of pennies must add an extra row and column to the current square, plus a corner penny where the new row and column meet. The row and column match exactly, making an even number, and then the extra penny at the corner makes it odd. Can you see that the "next odd number" pattern will continue as long as there are pennies to add? And that it could keep going forever in your imagination?

Twenty-five is a square number, because we can arrange twenty-five items to make a square: five rows with five items in each row.

The point of the penny square is not to memorize the square numbers or to get any particular "right answer," but to see numbers in a new way. To understand that numbers are related to each other. To realize we can show such relationships with diagrams or physical models. The more relationships like this our children explore, the more they see numbers as familiar friends.

A focus on answer-getting and test performance can ruin mathematics, distorting a discipline that is half art and half sport. Imagine a piano teacher who insisted her students spend six years on scales and exercises of gradually increasing difficulty before she would let them attempt a piece of actual music. Or a football coach who made his team run laps and do sit-ups every day, but let them play only two or three games a year, and scrimmage games at that. How many people would become bored with music or learn to hate football under such instruction?

As every coach knows, skill grows through practice. But practice has no meaning unless the team has a real game to play. And the best type of practice takes advantage of the benefits of cross-training by emphasizing variety rather than repetitive drills. Mathematical cross-training will include games, puzzles, stories, patterns, physical models, and the challenge of thinking things through.

Our children do need to learn how to perform routine calculations, as piano players must practice scales and football players lift weights. More important, however, our children need to learn why those operations work. And they must never be led to think that calculations are the essence of mathematics.

"A teacher of mathematics has a great opportunity," wrote Hungarian math professor George Polya. "If he fills his allotted time with drilling his students in routine operations he kills their interest, hampers their intellectual development, and misuses his opportunity.

"But if he challenges the curiosity of his students by setting them problems proportionate to their knowledge, and helps them to solve their problems with stimulating questions, he may give them a taste for, and some means of, independent thinking."

Playing with Numbers

Writing for *Family Life* magazine, mathematician and music critic Edward Rothstein described a game he invented for his daughter:

> *"What number am I? If you add me to myself, you get four."*

I gave that question to my six-year-old daughter during a family car trip. Then her sister, age nine, wanted in the game. I tried a question with bigger numbers, but she rolled her eyes. "That's too easy, Mom."

So I asked her:

> *"What number am I? If you take away one-fourth of me and then add two, you get seventeen." [answer1]*

The older our children get, the harder their parents have to work. For my twelve-year-old son, I asked:

"What number am I? If you multiply me by myself and add one, you get half as many as the number of pennies in a dollar." [answer2]

That kept him busy for a few minutes. After he figured it out, he came back with:

"What number am I? If you divide me by two and take away four, then add five, then multiply by three and divide by two and add seven, you get me again."

"What?" I asked. He repeated the question.

"This is actually a number?" I asked. "You figured out an answer to this?"

He nodded, with the smug grin of a preteen who knows he has Mom skewered.

I pulled out a notebook and pen. He repeated his series of calculations, and this time I wrote it down. I figured the answer had to be zero or one, those magic numbers that make multiplying easy, but neither worked.

I tried one hundred. No luck.

I heard a chuckle from the back seat.

"Wait," I said. "Give me a chance."

My husband was driving, but he glanced over at the notebook. "You know," he offered, "you could set that up as an equation."

Answers

You can find Answers to Sample Problems in the appendixes (page 241). Problem 1 has two possible solutions, depending on how you understand the words in the question. My daughter did not see it the same way I did. Her answer caught me by surprise—it was three times the number I expected—and yet after she explained her reasoning, I had to admit that her solution, too, was correct.

Let this be a warning: if your child's answer is not the same as yours, don't assume she is wrong. Ask her to explain how she figured it out. Then listen with care. Children almost always have a logical reason for their answers. Language is a complicated thing, and even a math problem may be open to different interpretations.

No way. The boy had not needed algebra to figure it out, so neither did I. So I tried ten, then fifty, then twenty. OK, that narrowed it down. Now I knew the answer had to be between twenty and fifty, but I had run out of easy numbers.

I nibbled on the end of my pen.

My son hummed to himself.

"I've got it." I spun around as far as the seat belt allowed. "The answer is …" [answer3]

"Nope."

"WHAT?"

I looked at my scratch paper. I worked the numbers again, coming up with the same answer. I read the steps of my calculation out loud.

He agreed that my number would work, but it was not the one he had in mind. I would have to guess again.

Hubby protested that there couldn't be another answer. If the equation doesn't have an x^2 or something similar, there can't be more than one solution.

The kid stood his ground, smirking.

I conceded. "What's your number?"

"Infinity. It doesn't matter what you multiply or take away, it's always infinity."

Aha! He was right. Well, sort of right: infinity isn't a real number, so you can't calculate with it that way. But it's good enough for middle school. Even better, he had a chance to stump the adults.

Playing with Shapes and Patterns

Even the simplest objects can provide an opportunity for mathematical play. The following story comes from math teacher Christopher Danielson, the primary organizer behind the hands-on "Math On-A-Stick" exhibit at the 2015 Minnesota State Fair.

Danielson writes …

I have spent the last four days playing and talking math with kids of all ages for eleven hours a day, paying close attention to how children behave in this space we've built. My number one message coming out of this work is *Let the children play.*

When children come to the egg table at Math On-A-Stick, they know right away what to do. There are brightly colored plastic eggs, and there are large, flat thirty-egg cartons. The eggs go in the cartons.

No one needs to give them instructions.

A typical three- or four-year old will fill the cartons haphazardly. She won't be concerned with the order she fills it, nor with the colors she uses, nor anything else. She'll just put eggs into the carton one at a time in a seemingly random order.

But when that kid plays a second or third time, emptying and filling her egg carton—without being told to do so—she usually begins to see new possibilities. After five or ten minutes of playing eggs, this child is filling the carton in rows or columns. Or she's making patterns such as pink, yellow, pink, yellow, and so on. Or she's counting the eggs as she puts them in the carton. Or she's orienting all of the eggs so they are pointy-side up.

The longer the child plays, the richer the mathematical activity she engages in. This is because the materials themselves have math built into them. The rows and columns of the egg crate; the colors and shape of the eggs; the fact that the eggs can separate into halves—all of these are mathematical features that kids notice and begin to play with as they spend time at the table.

We have seen four-year-olds spend an hour playing with the eggs.

I have observed that the children who receive the least instruction from parents, volunteers, or me are the most likely to persist. These are the children who will spend twenty minutes or more exploring the possibilities in the eggs.

The children who receive instructions from adults are least likely to persist. When a parent or volunteer says, "Make a pattern," kids are likely to do one of two things:

- Make a pattern, quit, and move to something else.
- Stop playing without making a pattern.

We adults have a responsibility to let the children play. We can be there to listen to their ideas as they do. We can play in parallel by getting our own egg cartons out and filling these cartons with our own ideas.

But when we tell kids to "make a pattern" or "use the colors," we are asking the children to fill their carton with our ideas, rather than allowing them to explore their own.

—Christopher Danielson

What Is Our Goal?

As children approach school age, today's parents face a bewildering array of choices. Many find it useful to write out (or at least to talk through) their educational ideals. Is our mission like filling the empty bucket of a child's mind, or is it more like lighting a fire that will grow and spread on its own? Or is our role not to "teach" at all, but rather to walk alongside and assist our children as they explore the world? How we define our goals will make an enormous difference in how we approach the day by day adventure of learning.

In the same way, before we can figure out how to help our children with math, we need to think about our goals. What does it take to understand mathematics? Is it truly necessary for our children? After all, computers and calculators crunch most of the numbers in our modern world. Some children must grow up to program those computers, but what if my kids have other plans?

Do I want my children to learn math only because the state requires it? The state requires our children to learn math so they will be functionally literate. That may not sound like a lofty goal, but think about what "functionally literate in math" means:

- ♦ Filling out an IRS Form 1040 with its Schedule A, Schedule B, Schedule SE, and all the rest.
- ♦ Reading a mortgage and understanding how a fixed- or variable-rate loan will affect family finances.
- ♦ Following newspaper articles about the governor's budget proposal or discerning the relevance of political polls.
- ♦ Knowing that a 40% chance of rain on Saturday and a 60% chance of rain on Sunday doesn't mean there is a 100% chance of rain this weekend.

Mathematical literacy is a worthy challenge. But most of us want more than literacy for our children. We want them to be educated. An educated person is interested in more than merely what is useful. He or she loves to learn, studies for the sake of gaining knowledge, and grows in wisdom.

Few people read Shakespeare because his plays are useful. But an educated person will enjoy Shakespeare because his stories are interesting and his dialogue insightful. Likewise, much of math does not seem to be useful, but it can be fascinating.

If you want to experience the joy and artistry of math, consider *fractals*, which are intricate patterns of regular irregularity that capture something of the complex beauty in nature. The men who first discovered fractals in the late nineteenth century called them "monster curves." They could not imagine any use for such absurdities. Yet today, these monsters are used to compress images and other data files and have become a staple tool in film makers' special-effects kit.

Children who are educated in math will gain practical skills. But what is more important, those who enjoy learning for its own sake will find plenty to fascinate them.

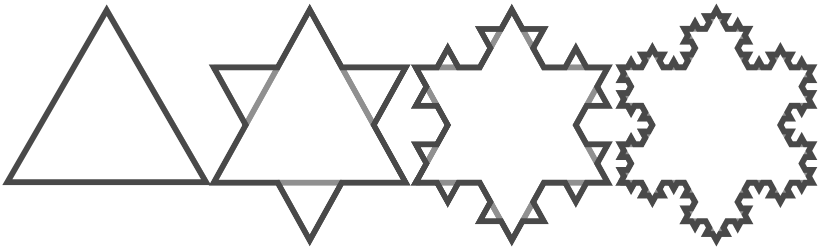
Math the Mathematician's Way

> *Real mathematics is not just formulaic tutoring. My hope is that children learn to think about mathematics as a kind of mental play.*
>
> —Edward Rothstein

Mathematics is mental play, the essence of creative problem solving. This is the truth we need to impart to our children, more important than fractions or decimals or even the times tables. Math is a game, playing with ideas.

Traditional school math is a lock-step sequence of topics, but math the mathematician's way is a social adventure of exploring and sharing new ideas. Math the mathematician's way is fun. It can even be beautiful. Listen to how real mathematicians, both professionals and amateurs who enjoy working with math, describe their subject:

> *I love mathematics principally because it is beautiful, because man has breathed his spirit of play into it, and because it has given him his*



Step by step, the boundary of the Koch Island fractal—sometimes called the Koch Snowflake—approaches infinite raggedness, much as an actual coastline appears more and more jagged under increasing magnification.

By allowing random variation as they design a fractal, artists can create natural-looking forms such as this mountainous coastline in winter.

greatest game—the encompassing of the infinite.

—Rózsa Péter

There is no ulterior practical purpose here. I'm just playing. That's what math is: wondering, playing, amusing yourself with your imagination.

—Paul Lockhart

Puzzles in one sense, better than any other single branch of mathematics, reflect its always youthful, unspoiled, and inquiring spirit. When a man stops wondering and asking and playing, he is through.

—Edward Kasner

If mathematics education communicated this playful aspect of the subject, I don't think innumeracy would be as widespread as it is.

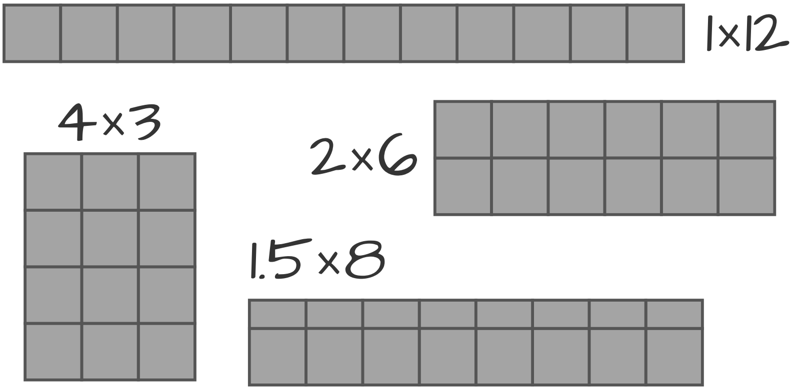
—John Allen Paulos

W. W. Sawyer wrote a book called *Mathematician's Delight,* in which he described mathematical thinking this way: "Everyone knows that it is easy to do a puzzle if someone has told you the answer. That is simply a test of memory. You can claim to be a mathematician only if you can solve puzzles that you have never studied before. That is the test of reasoning."

It is also the test of life. Every day we face puzzles we have not studied before. Math taught the mathematician's way prepares us to approach problems with confidence. It teaches us to see our mistakes as stepping stones to learning. It reminds us that there may be more than one right answer, as my children and I discovered when we played "What Number Am I?"

Math taught the mathematician's way gives our children practice struggling with challenging problems. It lets them enjoy that "Aha!" thrill when they find a solution. Math the mathematician's way prepares them for careers or college. It gives them tools they can use throughout their lives. It gives them confidence by letting them succeed at something difficult. When kids solve a puzzle that stumped their parents, they know they can handle anything.

Children who play around with math taught the mathematician's way may not follow the schedule demanded by their state's math standards. If we de-emphasize worksheets, timed drills, and anxiety-producing tests, we can encourage our children to focus on reasoning skills and thinking



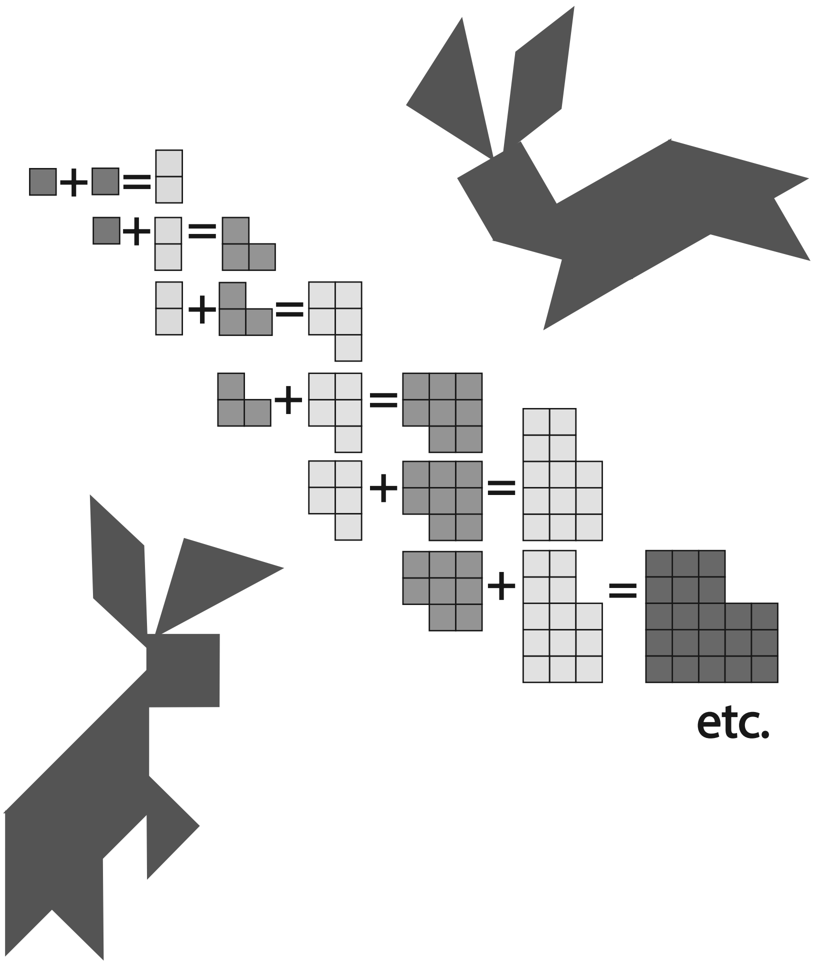
Math with many right answers: how long are the sides of a rectangle with an area of twelve squares?

a situation through. We don't need to quiz our kids on the addition facts or times tables.

Of course our children need to know how to add and subtract, multiply and divide. But wonder-inducing ideas such as Fibonacci numbers (see Fibonacci's Rabbit Problem on page 130) give students an adventure in real-world addition that goes far beyond the basic math facts. Playing around with exponential growth (see the Penny Birthday Challenge on page 68) provides experience with real-world multiplication in a way that even many adults fail to understand.

Flashcard drills and practice pages of arithmetic problems are the least effective ways to learn math. Rote learning is the laziest, most deadening way to approach any subject. Think of memorizing dates in history class, and compare that to reading historic speeches and well-written biographies. On occasion, such memory work may be necessary, but it should never be the bulk of a child's experience in any subject.

Instead, we need to introduce our students to the thrill of tackling tough, challenging puzzles. We need to give children a taste of the joy that comes from figuring things out, the "Aha!" factor. We need to adopt the mathematician's view of math as mental play. Learning to think a problem through can be hard work—and that is exactly what makes it fun.



In the late 12th century, Leonardo of Pisa—also known as Fibonacci—created a small story problem about breeding rabbits, which leads to the Fibonacci number sequence: 1, 1, 2, 3, 5, 8, 13, 21 … The rabbits in the picture are tangram puzzles (see page 56).

2

Most adults, including teachers, mistake showing students how to do a calculation with teaching them to understand math.

—Rick Garlikov

Think Like a Mathematician

All parents and teachers have one thing in common: we want our children to understand and be able to use math. Counting, multiplication, fractions, geometry—these topics are older than the pyramids. So why is mathematical mastery so elusive?

"The root problem is that we're all graduates of the same system," says math educator Burt Furuta. "The vast majority of us, including those with the power to shape reform, believe that if we can compute the answer, then we understand the concept; and if we can solve routine problems, then we have developed problem-solving skills."

The culture we grew up in, with all its strengths and faults, shaped our perception of math, as we in turn shape the experience of our children.

Like any human endeavor, American math education—the system I grew up in—suffers from a series of fads. I lived through the experiment with hyper-abstract New Math in the 1960s. That led to the reactionary Back to Basics movement of the 1970s and 80s. In the last part of the twentieth century, Reform Math focused on problem solving, discovery learning, and student-centered methods. But Reform Math brought calculators into elementary classrooms and de-emphasized pencil-and-paper arithmetic, setting off a "Math War" with those who argued for a more traditional approach.

At this writing, policymakers in the U.S. are debating the Common Core State Standards initiative. These guidelines attempt to blend the best parts of reform and traditional mathematics. They try to balance an emphasis on conceptual knowledge with development of procedural fluency. The "Standards for Mathematical Practice" encourage us to make sense of math problems and persevere in solving them, to give explanations for our answers, and to listen to the reasoning of others—all of which are important aspects of math as mental play.† But the rigid way in which the Common Core standards were imposed and the ever-increasing emphasis on standardized tests seem likely to sabotage any hope of peace in the Math Wars.

Through all the math education fads, however, one thing remains consistent. Even before they reach the schoolhouse door, children are convinced that math is all about memorizing and following arbitrary rules. Understanding math, according to popular culture—according to movie actors, TV comedians, politicians pushing "accountability," and the aunt who quizzes you on your times tables at a family gathering—means knowing which procedures to apply so you can get the correct answers.

But when mathematicians talk about understanding math, they have something different in mind. To them, mathematics is all about ideas and the relationships between them. Understanding math means seeing the patterns in these relationships: how things are connected, how they work together, and how a single change can send ripples through the system.

What Is Your Mathematical Worldview?

Educational psychologist Richard Skemp popularized the terms *instrumental understanding* and *relational understanding* to describe these two ways of looking at mathematics. It is almost as if there were two unrelated subjects. Both are called "math," but they are different from each other as American football is from the game the rest of the world calls football.

Which of the following sounds the most like your experience of school math? Which type of math are your children learning?

† *http://www.corestandards.org/Math/Practice*

Instrumental Understanding: Math as a Tool

Every mathematical procedure we learn is an instrument or tool for solving a certain kind of problem. To understand math means to know which tool we are supposed to use for each type of problem and how to use that tool—how to categorize the problem, remember the formula, plug in the numbers, and do the calculation. To be fluent in math means we can produce correct answers with minimal effort.

Primary goal: to get the right answer. In math, answers are either right or wrong, and wrong answers are useless.

Key question: "What?" What do we know? What can we do? What is the answer?

Values: speed and accuracy.

Method: memorization. Memorize math facts. Memorize definitions and rules. Memorize procedures and when to use them. Use manipulatives and mnemonics to aid memorization.

Benefit: testability.

Instrumental instruction focuses on the standard *algorithms* (the pencil-and-paper steps for doing a calculation) or other step-by-step procedures. This produces quick results because students can follow the teacher's directions and crank out a page of correct answers. Students like completing their assignments with minimal struggle. Parents are pleased by their children's high grades. Teachers are happy to make steady progress through the curriculum.

Unfortunately, the focus on rules can lead children to conclude that math is arbitrary and authoritarian. Also, rote knowledge tends to be fragile. All those steps are easy to confuse or forget. Thus those who see math instrumentally must include continual review of old topics and provide frequent, repetitive practice.

Relational Understanding: Math as a Connected System

Each mathematical concept is part of a web of interrelated ideas. To understand mathematics means to see at least some of this web and to use the connections we see to make sense of new ideas. Giving a correct answer without *justification* (explaining how we know it is right) is mere accounting, not mathematics. To be fluent in math means we can think of more than one way to solve a problem.

Primary goal: to see the building blocks of each topic and how that topic relates to other concepts.

Key questions: "How?" and "Why?" How can we figure that out? Why do we think this is true?

Values: logic and justification.

Method: conversation. Talk about the links between ideas, definitions, and rules. Explain why you used a certain procedure, and explore alternative approaches. Use manipulatives to investigate the logic behind a technique.

Benefit: flexibility.

Relational instruction focuses on children's thinking and expands on their ideas. This builds the students' ability to reason logically and to approach new problems with confidence. Mistakes are not a mark of failure, but a sign that points out something we haven't yet mastered, a chance to reexamine the mathematical web. Students look forward to the "Aha!" feeling when they figure out a new concept. Such an attitude establishes a secure foundation for future learning.

Unfortunately, this approach takes time. It requires extensive personal interaction. Discussing problems. Comparing thoughts. Searching for alternate solutions. Hashing out ideas. Those who see math relationally must plan on covering fewer new topics each year, so they can spend the necessary time to draw out and explore these connections. Relational understanding is also much more difficult to assess with a standardized test.

Is There Really a Difference?

From the outside, it's impossible to tell how a person is thinking. A boy with the instrumental perspective and a girl who reasons relationally may both get the same answers on a test. Yet under the surface, in their thoughts and how they view the world, they could not be more different.

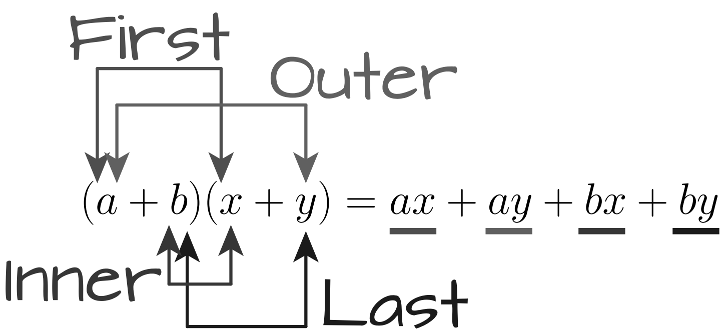
"Mathematical thinking is more than being able to do arithmetic or solve algebra problems," says Stanford University mathematician and popular author Keith Devlin. "Mathematical thinking is a whole way of looking at things, of stripping them down to their numerical, structural, or logical essentials, and of analyzing the underlying patterns."

Our mathematical worldview influences the way we present math topics to our kids. Consider, for example, the following three rules that most of us learned in middle school.

- ♦ Area of a rectangle = length × width.
- ♦ To multiply fractions, multiply the tops (numerators) to make the top of your answer. Multiply the bottoms (denominators) to make the bottom of your answer.

$$\frac{3}{4} \times \frac{5}{8} = \frac{(3 \times 5)}{(4 \times 8)} = \frac{15}{32}$$

- ♦ When you need to multiply algebra expressions, remember to FOIL. Multiply the First terms in each parenthesis. Then multiply the Outer terms. Then the Inner and Last pairs. Finally, add all those answers together.



While the times symbol or the word *multiply* is used in each of these situations, the procedures are completely different. How can we help our children understand and remember these rules?

Over the next several pages, we'll dig deeper into each of these math rules as we examine what it means to develop relational understanding.

Many people misunderstand the distinction between instrumental and relational understanding. They think the terms refer to surface-level, visible differences in instructional approach, but it's not that at all. It has nothing to do with our parenting or teaching style, or whether our kids are learning with a traditional textbook or through hands-on projects. It's not about using "real world" problems, except to the degree that the world around us feeds our imagination and gives us the ability to think about math concepts.

This dichotomy is all about the vision we have for our children—what we imagine mathematical success to look like. That vision may sit below the level of conscious thought, yet it shapes everything we do with math. And our children's vision for themselves shapes what they pay attention to, care about, and remember.

Area of a Rectangle

The instrumental approach to explaining such rules is for the adult to work through a few sample problems and then give the students several more for practice. In a traditional lecture-and-workbook style curriculum, students apply the formula to drawings on paper. Under a more progressive reform-style program, the students may try to invent their own methods before the teacher provides the standard rule, or they may measure and calculate real-world areas such as the surface of their desks or the floor of their room. Either way, the ultimate goal is to define terms and master the formula as a tool to calculate answers.

Richard Skemp describes a typical lesson:

> *Suppose that a teacher reminds a class that the area of a rectangle is given by A=L×B. A pupil who has been away says he does not understand, so the teacher gives him an explanation along these lines. "The formula tells you that to get the area of a rectangle, you multiply the length by the breadth."*
>
> *"Oh, I see," says the child, and gets on with the exercise.*
>
> *If we were now to say to him (in effect) "You may think you understand, but you don't really," he would not agree. "Of course I do.*

Look; I've got all these answers right."

Nor would he be pleased at our devaluing of his achievement. And with his meaning of the word, he does understand.

As the lesson moves along, students will learn additional rules. For instance, if a rectangle's length is given in meters and the width in centimeters, we must convert them both to the same units before we calculate the area. Also, our answer will not have the same units as our original lengths, but that unit with a little, floating "2" after it, which we call "squared." Each lesson may be followed by a section on word problems, so the students can apply their newly learned rules to real-life situations.

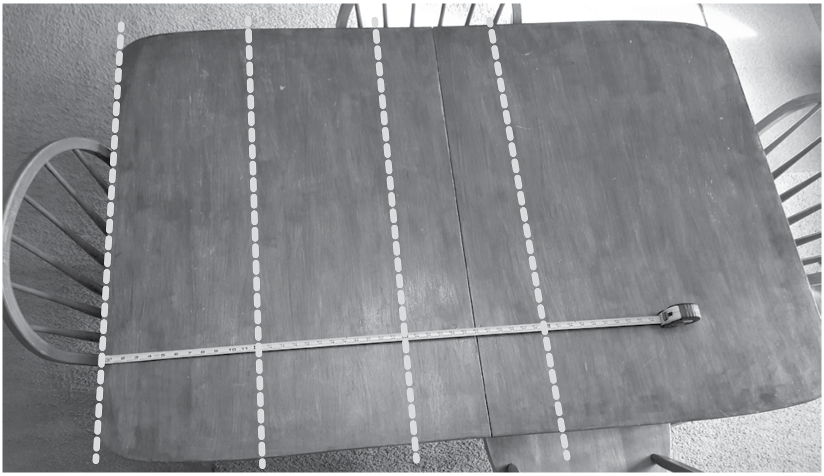
In contrast, a relational approach to area must begin long before the lesson on rectangles. Again, this can happen in a traditional, teacher-focused classroom or in a progressive, student-oriented environment. Either way, the emphasis is on uncovering and investigating the conceptual connections that lie under the surface and support the rules.

We start by exploring the concept of measurement. Our children measure a path along the floor, sidewalk, or anywhere we could imagine moving in a straight line. We learn to add and subtract such distances. Even if our path turns a corner or if we first walk forward and then double back, it's easy to figure out how far we have gone.

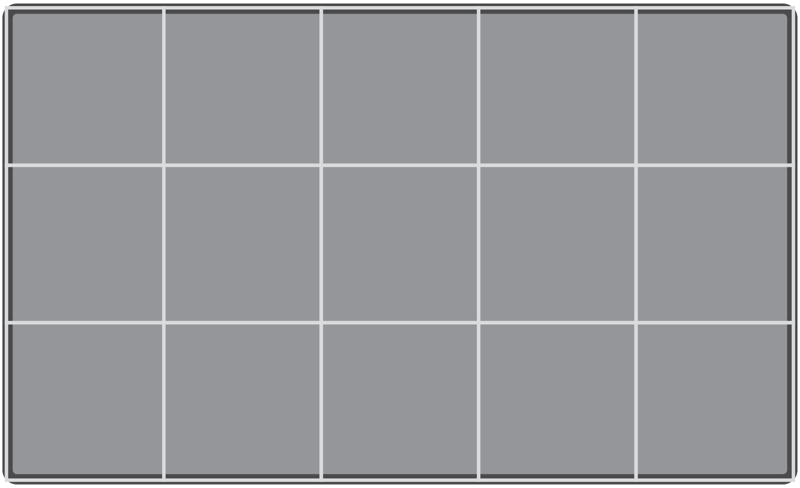
But something strange happens when we consider distances in two different directions at the same time. Measuring the length and width of the dining table automatically creates an invisible grid. In measuring the length of a rectangular table, we do not find just one point at any given distance. There is a whole line of points that are one foot, two feet, or three feet from the left side of the table.

And measuring the width shows us all the points that are one, two, or three feet from the near edge. Now our rectangular table is covered by virtual graph paper with squares the size of our measuring unit.

The length of the rectangle tells us how many squares we have in each row. The width tells us how many rows there are. As we imagine this invisible grid, we can see why multiplying those two numbers will tell us how many squares there are in all. That is what the word *area* means. The area of a tabletop is the number of virtual-graph-paper squares it takes to cover it up, which is why our answer will be measured in square units.

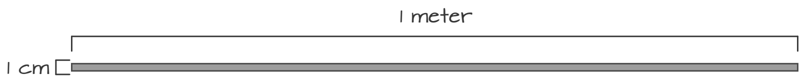


Measuring the distance from one edge of a table. Apologies to my metric-speaking readers, but the old-fashioned foot is the most convenient unit for this demonstration.



The rectangular tabletop with an imaginary grid that shows the length and width measurements: three feet wide by five feet long.

What if we measured the length in meters and the width in centimeters? With a relational understanding of area, even a strange combination of units can make sense. Our invisible grid would no longer consist of squares but of long, thin, rectangular centimeter-meters. But we could still find the area of the tabletop by counting how many of these units it takes to cover it. Square units aren't magic—they're just easier, that's all.



How many rectangles will we need to cover a table that is 2 m long by 90 cm wide? $2 \times 90 = 180$ centimeter-meters.

Multiplying Fractions

Fractions inspire more math phobia among children (and adults) than any other topic before algebra. Children begin learning fractions by coloring or cutting up paper shapes. Their intuition is shaped by experiences with food like sandwiches or pizza. But before long, the abstraction of written calculations looms up to swallow intuitive understanding.

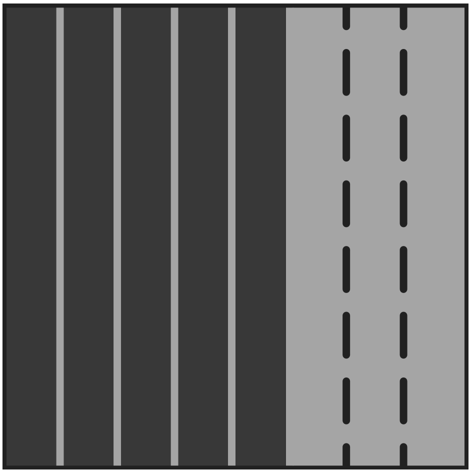
Upper elementary and middle school classrooms devote many hours to working with fractions, and still students flounder. In desperation, parents and teachers resort to nonsensical mnemonic rhymes that just might stick in a child's mind long enough to pass the test.

How can we make sense of the fraction rules?

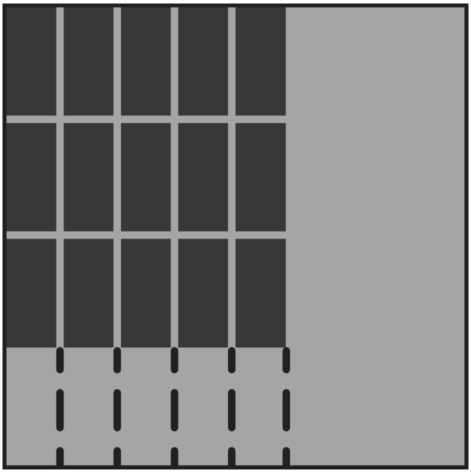
Let's go back to our rectangular tabletop, and this time we'll zoom in. Imagine magnifying our virtual grid to show a close-up of a single square unit, such as the pan of brownies on our table. We can imagine subdividing this square into smaller, fractional pieces. In this way, we see that five-eighths of a square unit looks something like a pan of brownies cut into strips, with a few strips missing:

But what if we don't even have that whole five-eighths of the pan? Imagine that the kids came through the kitchen and snatched a few more pieces. Now all we have is three-fourths of the five-eighths.

So three-fourths of five-eighths is a small rectangle of single-serving pieces. How much of the original pan of brownies do we have now?



One batch of brownies is one square unit, but part of the batch has been eaten. Now we have fractional brownies: five-eighths of the pan.

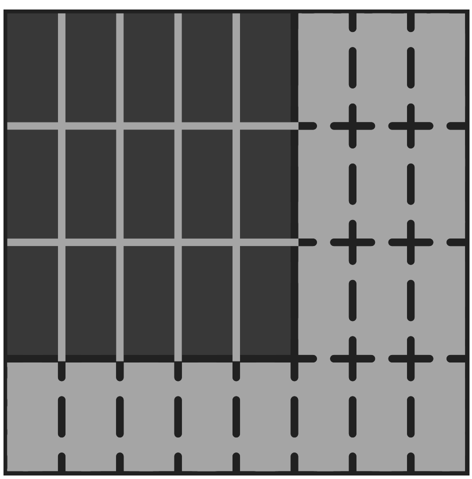


3/4 of 5/8. We can make a fraction of a fraction by cutting the other direction. In this case, we cut the strips into fourths, and the kids ate one part of each strip.

There are three rows with five pieces in each row, for a total of 3 × 5 = 15 pieces left—which is the numerator of our answer. With pieces that size, it would take four rows with eight in each row (4 × 8 = 32) to fill the whole pan—which is our denominator, the number of pieces in the whole batch of brownies.

Notice that there was nothing special about the fractions 3/4 and 5/8, except that the numbers were small enough for easy illustration. We could imagine a similar pan-of-brownies approach to any fraction multiplication problem, though the final pieces might turn out to be crumbs.

Children won't draw brownie-pan pictures for every fraction multiplication problem the rest of their lives. But they do need to spend plenty of time thinking about what it means to take a fraction of a fraction and how that meaning controls the numbers in their calculation. They need to ask questions, model situations, put things in their own words, and wrestle with the concept until it makes sense to them. Only then will their understanding be strong enough to support future learning.



Compare the pieces we have left to the original batch. Each of the numbers in the fraction calculation has meaning. Can you find them all in the picture?

$$\frac{3}{4} \times \frac{5}{8} = \frac{(3 \times 5)}{(4 \times 8)} = \frac{15}{32}$$

Algebraic Multiplication

Let's extend the rectangle concept even further. In the connected system of mathematics, almost any type of multiplication can be imagined as a rectangular area. We don't even have to know the size of our rectangle. It could be anything, such as subdividing a plot of land or designing a section of crisscrossed colors on plaid fabric.

We can imagine a rectangle with each side made up of two unknown lengths. One side has some length a attached to another length b. The other side is x units long, with an extra amount y stuck to its end.

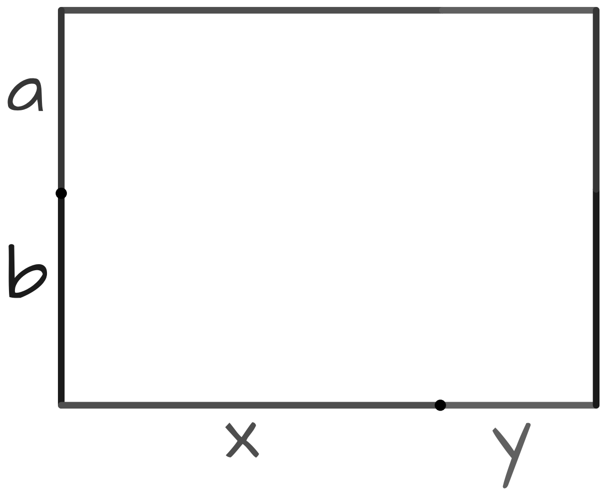
We don't know which side is the "length" or which is the "width" because we don't know which numbers the letters represent. But multiplication works in any order, so it doesn't matter which side is longer. Using the rectangle model of multiplication, we can see that this whole shape represents the area $(a + b)(x + y)$.

But since each side is measured in two parts, we can also imagine cutting up the big rectangle. The large, original rectangle covers the same amount of area as the four smaller pieces added together. Thus we can see that $(a + b)(x + y) = ax + ay + bx + by$.

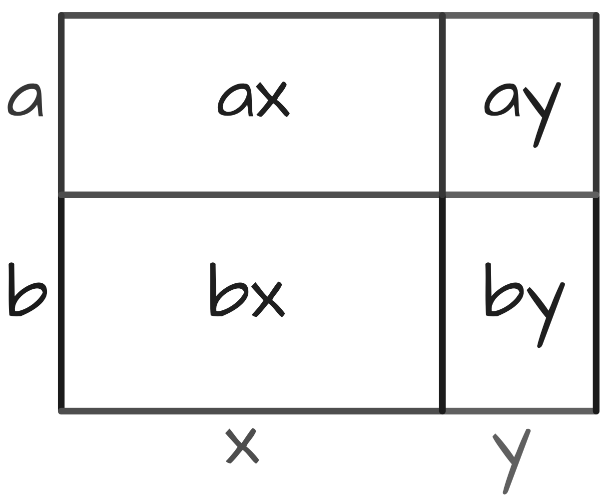
At the beginning of this section, I gave you three middle-school math rules. But by exploring the concept of rectangular area as a model of multiplication, we discover that in a way they were all the same. The rules are not arbitrary, handed down from a mathematical Mount Olympus. They are three expressions of a single basic question: what does it mean to measure area? There was only one rule, one foundational pattern that tied all these topics together in a mathematical web.

Many children want to learn math instrumentally, as a tool for getting answers. They prefer the simplicity of memorizing rules to the more difficult task of making sense of new ideas. Being young, they are by nature short-term thinkers. They beg, "Just tell me what to do."

But if we want our children to understand math the mathematician's way, we need to resist such shortcuts. We must take time to explore mathematics as a world of ideas that connect and relate to each other in many ways. We need to show children how to reason about these interconnected concepts, so they can use them to think their way through an ever-expanding variety of problems.



An algebraic rectangle: each side is composed of
two unknown lengths joined together.

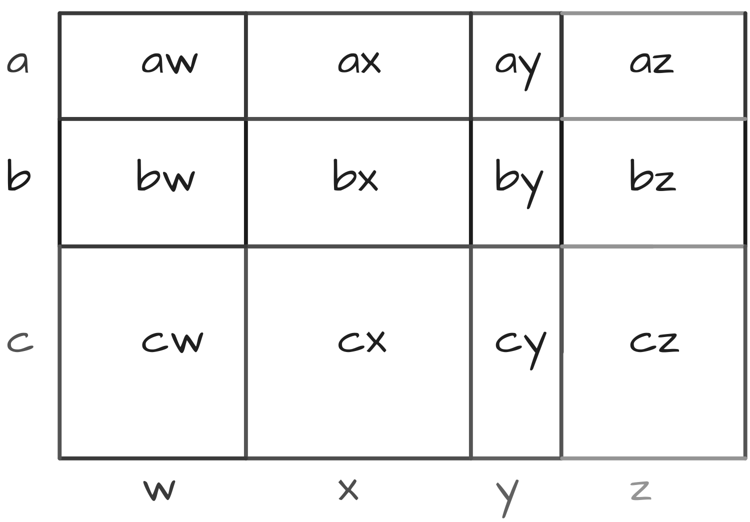


Four algebraic rectangles: the whole thing
is equal to the sum of its parts.

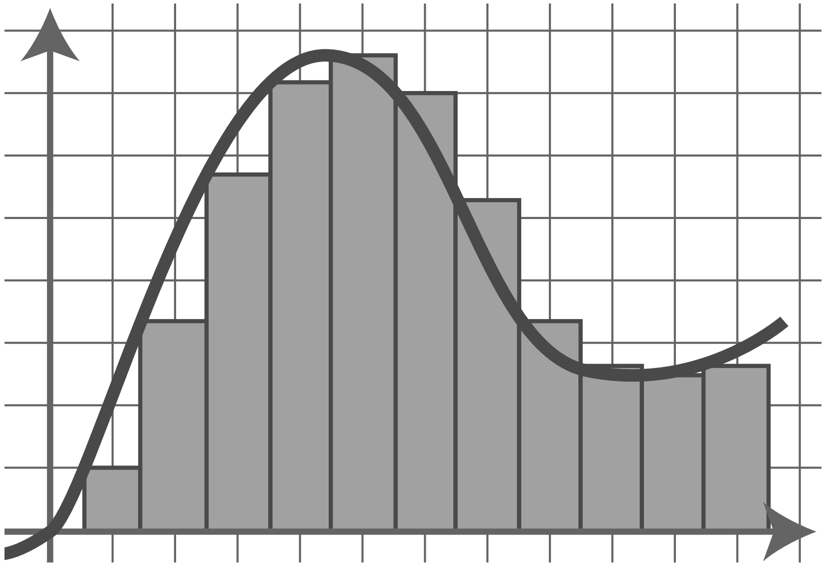
With the memorized FOIL formula mentioned earlier, for example, our students may get a correct answer quickly, but it's a dead end. FOIL doesn't connect to any other math concepts, not even other forms of algebraic multiplication. But the rectangular area model will help our kids multiply more complicated algebraic expressions such as $(a + b + c)$ times $(w + x + y + z)$.

Not only that, but the rectangle model gives students a tool for making sense of later topics such as polynomial division. And it is fundamental to understanding integral calculus.

Our kids can only see the short term. If we adults hope to help them learn math, our primary challenge is to guard against viewing the mastery of facts and procedures as an end in itself. We must never fall into thinking that the point of studying something is just to get the right answers. We understand this in other school subjects. Nobody imagines that the point of reading is to answer comprehension questions. We know that there is more to learning history than winning a game of Trivial Pursuit. But when it comes to math, too many parents (and far too many politicians) act as though the goal of our children's education is to produce high scores on a standardized test.



The rectangle model of multiplication helps students keep track of all the pieces in a complex algebraic calculation.



In calculus, students use the rectangle model of multiplication to find irregular areas. The narrower the rectangles, the more accurate the calculation, so we imagine shrinking the widths until they are infinitely thin.

What If I Don't Understand Math?

If you grew up (as I did) thinking of math as a tool, the instrumental approach may feel natural to you. The idea of math as a cohesive system may feel intimidating. How can we parents help our children learn math, if we never understood it this way ourselves?

Don't panic. Changing our worldview is never easy, yet even parents who suffer from math anxiety can learn to enjoy math with their children. All it takes is a bit of self-discipline and the willingness to try. You don't have to know all the answers. In fact, many people have found the same thing that Christopher Danielson described in Chapter 1 (page 15)—the more we adults tell about a topic, the less our children learn. With the best of intentions we provide information, but we unwittingly kill their curiosity.

If you're afraid of math, be careful to never let a discouraging word pass your lips. Call upon your acting skills to pretend that math is the most exciting topic in the world. Encourage your children to notice the math all around them. Investigate, experiment, estimate, explore, measure—and talk about it all. Curl up together on the couch to read a math book, or tell math stories at bedtime. Search out opportunities to discuss numbers, shapes, symmetry, and patterns with your kids.

Patterns are so important that American mathematician Lynn Arthur Steen defined mathematics as *the science of patterns.*

"As biology is the science of life and physics the science of energy and matter, so mathematics is the science of patterns," Steen wrote. "We live in an environment steeped in patterns—patterns of numbers and space, of science and art, of computation and imagination. Patterns permeate the learning of mathematics, beginning when children learn the rhythm of counting and continuing through times tables all the way to fractals and binomial coefficients."

If you are intimidated by numbers, you can look for patterns of shape and color. Pay attention to how they grow. Talk about what your children notice. For example, some patterns repeat exactly, while other patterns change as they go (small, smaller, smallest, or loud, louder, loudest). Nature often forms fractal-like patterns—the puffy round-upon-roundness of cumulus clouds or broccoli, or the branch-upon-branchiness of a shrub or river delta. Children can learn to recognize these, not as a homework exercise but because they are interesting.

Here is the secret solution to the crisis of math education: we adults need to learn how to think like mathematicians.

- Mathematicians avoid busywork as if it were an infectious disease.
- Mathematicians always ask questions.
- Most of all, mathematicians love to play.

As we cultivate these characteristics, we will help our children to recognize and learn true mathematics.

Mathematicians Avoid Busywork

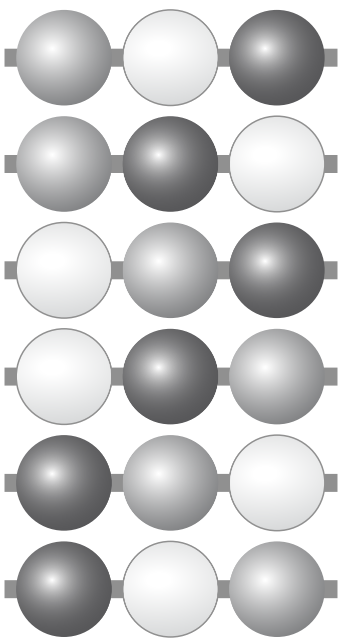
Mathematicians are economical with their time and energy. You might even say they are "lazy." They have too many interesting topics to study and not enough time to learn about them all, so they cannot afford to waste their time on mindless busywork.

Mathematicians are always looking for a simpler way to do things. Skip counting is faster than addition; multiplication is even easier. Algebraic functions are shorthand ways to describe how things grow and change. Calculus lets engineers solve problems that would be impossible without it.

Some problems have more potential solutions than anyone could list, so mathematicians invented special ways to think about counting. *Permutations* let us count the possibilities when the order matters: "How many different ways might we award the first, second, and third place prizes in an art show?" *Combinations* count the ways to do something when order doesn't matter: "How many different three-person committees might we choose to plan the party for our neighborhood co-op?" Both types of puzzle can be interesting for middle school or older students.

When we look at education through the lens of mathematical laziness, we will be skeptical about the value of repetitive practice problems. If our children keep forgetting how to do certain calculations, perhaps they are not yet developmentally ready to learn them. One of the biggest problems in math education is how often we train kids to do "number tricks" the same way we would train a dog to fetch or roll over, by having them repeat the procedure until they can do it without thinking. Children will master math better if we wait until they have developed a foundation that will help them understand why it works.

In the meantime, there are ever so many interesting ideas we can explore together. Parents who think like mathematicians will always make time for the fun stuff.



How many ways can you arrange one red, one yellow, and one blue bead on a string? When the order matters, you are counting permutations.

Mathematicians Ask Questions

Wise mathematicians are never satisfied with finding the answer to a problem. If they decide to put effort into solving any math puzzle, then they are determined to milk every drop of knowledge they can get from that problem. When mathematicians find an answer, they always go back and think about the problem again. Is there another way to look at it? Can we make our solution simpler or more elegant? Does this problem relate to any other mathematical idea? Can we expand our solution? Can we find a general principle?

As math teacher Herb Gross says, "What's really neat about mathematics is that even when there's only one right answer, there's never only one right way to do the problem."

School textbooks ask questions for which they know the answer. As we learn to think like mathematicians, we will ask a different type of question. Asking questions to which we don't know the answer can be frightening, because it forces us to give up the illusion of being in control. It makes us vulnerable, but it opens the door to learning.

- What do you think?
- What do you notice?
- What does it remind you of?
- What do you wonder?
- Is there another way to look at it?
- Will this always be true?
- If it's only true sometimes, what are the conditions that make it true? What conditions make it false?
- Could part of it be true and part of it be false?
- If this is false, then is something else true?
- Can you predict what will happen next?
- How did you figure that out?
- Is there a pattern?
- Will the pattern continue, or will it run out?

♦ How can we be sure?

♦ How would you change it?

♦ What would happen if ___?

♦ Why?

You can try asking your children questions to which you don't know the answer. Or encourage your kids to take the lead: "What questions can we ask about this?" When children learn to pose questions, that's when they start thinking like mathematicians.

As your children try to put their thoughts into words, keep in mind this truth:

> *Most remarks made by children consist of correct ideas very badly expressed. A good teacher will be wary of saying "No, that's wrong." Rather, he will try to discover the correct idea behind the inadequate expression. This is one of the most important principles in the whole of the art of teaching.*
>
> —W. W. Sawyer

Don't worry if you can't find the answers to all the questions you or your children ask. Some mathematical questions took centuries to answer and led to new branches of study, while others are still open. In the quest of learning math, wondering can be its own reward.

Mathematicians Love to Play

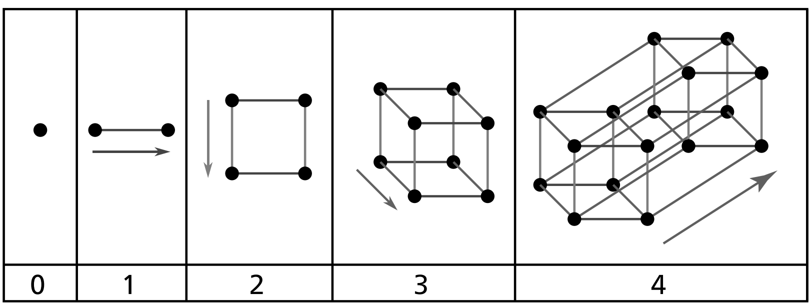
> *The mathematician plays a game in which he himself invents the rules while the physicist plays a game in which the rules are provided by nature, but as time goes on it becomes increasingly evident that the rules which the mathematician finds interesting are the same as those which nature has chosen.*
>
> — Paul Dirac

Mathematicians play with ideas. They toy with puzzles. They tinker with the connections between shapes and numbers, mystery and logic, growth and change. To a mathematician, the fun of the game is in experimenting, in trying new things and discovering what will happen. Many modern strategy games were invented for the puzzle of analyzing who would win.

For example, consider the simplest form of the two-player strategy game Nim. In this game, you start with a pile of pebbles or other small items. Take turns removing either one or two stones from the pile until one player is forced to take the last one, thus losing the game. Play the game with your children several times, and then encourage them to think of some way to change the rules. Ask questions: How does the game change? Is the new version easier or harder to win? Does the player who goes first have an advantage?

Parents who think like mathematicians will join their children in playing with numbers, shapes, and patterns. We might cut a sandwich in half, and then cut the halves in half, and then cut the half-halves in half, and see how small we can go. Or we may build a perfect square pyramid out of sugar cubes (with 1, 4, 9, 16, 25, 36, and 49 cubes in the layers) as the centerpiece for our next tea party.

Or we could think about new ideas: a point has no dimension, no length or width or size at all. But if we imagine pulling hard to stretch it out in one direction, it would become a one-dimensional line. A line pulled out and stretched into two dimensions becomes a square. A square

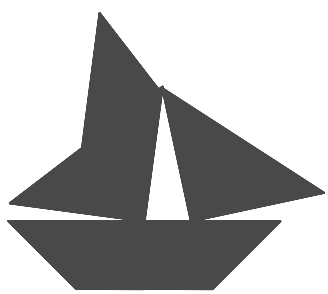


The dimension of an object is the number of ways a point can move on that object. On a line, for instance, a point can only move sideways, so a line is one-dimensional. On a sheet of paper, however, a point could move sideways or up-and-down (or some combination of those two motions), so a flat shape is two-dimensional. Because we live in a 3-D world, we can easily imagine shapes up to three dimensions, but higher dimensions seem strange and confusing. Some physicists speculate that the universe may have 10 or more dimensions, most of which are invisible to us.

stretched into three dimensions becomes a cube. So what would happen if we could stretch a cube into four dimensions?

Please don't be like the algebra teacher who caught a student playing tic-tac-toe and snatched away his paper, saying, "When you're in my classroom I expect you to work on mathematics." When the boy's friend, popular math writer Martin Gardner, heard the story, he responded that tic-tac-toe was an excellent introduction to symmetry, probability, set theory, multi-dimensional geometry, and other advanced topics. "With a little guidance," Gardner said, "it might have been much more rewarding than what his teacher was teaching."

Learning to think like a mathematician is a lifetime adventure. I don't have room in this book to mention perfect numbers, spherical geometry, Penrose tiles, cryptograms, or any of a zillion other topics waiting to be explored. When you begin to look at the world with a mathematician's eye, you embark on a journey more varied than the voyage of Ulysses. More exotic than Marco Polo's travels. More adventuresome than a trip to the moon.



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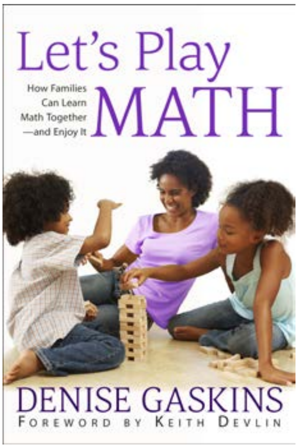

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Let's Play Math:

How Families Can Learn Math Together —and Enjoy It

All parents and teachers have one thing in common: we want our children to understand and be able to use math. Counting, multiplication, fractions, geometry—these topics are older than the pyramids. So why is mathematical mastery so elusive?

Written by a veteran homeschooling mom, *Let's Play Math* offers a wealth of practical, hands-on ideas for exploring math from preschool to high school. Whether you want to balance and enrich a traditional curriculum or launch an off-road mathematical adventure of your own, this book helps you:

- ♦ Introduce your children to the "Aha!" factor—the thrill of conquering a tough challenge.
- ♦ Discover activities that will awaken your children's minds to the beauty and fun of mathematics.
- ♦ Build thinking skills with toys, games, and library books.
- ♦ Find out how to choose math manipulatives or to make your own.
- ♦ And learn how to tackle story problems with confidence.

True mathematical thinking involves the same creative reasoning that children use to solve puzzles. Your children will build a stronger foundation of understanding when you approach math as a family game, playing with ideas.

About the Author

DENISE GASKINS ENJOYS MATH, AND she delights in sharing that joy with young people. "Math is not just rules and rote memory," she says. "Math is like ice cream, with more flavors than you can imagine. And if all you ever do is textbook math, that's like eating broccoli-flavored ice cream."

A veteran homeschooling mother of five, Denise has taught or tutored mathematics at every level from pre-K to undergraduate physics. "Which," she explains, "at least in the recitation class that I taught, was just one story problem after another. What fun!"

Now she writes the popular blog Let's Play Math and manages the Math Teachers at Play monthly math education blog carnival.

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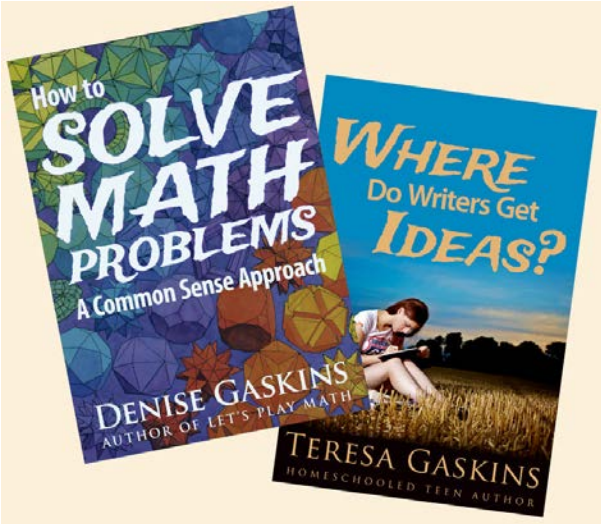

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Are you tired of the daily homework drama? Do your children sigh, fidget, whine, stare out the window—anything except work on their math? Wouldn't it be wonderful if math was something your kids wanted to do?

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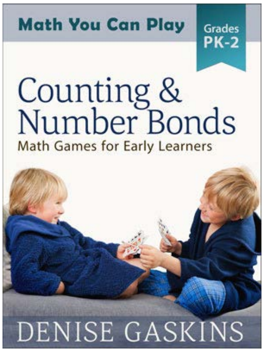
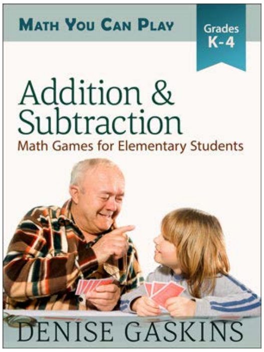
Math games pump up mental muscle, reduce the fear of failure, and develop a positive attitude toward mathematics. Through playful interaction, games strengthen a child's intuitive understanding of numbers and build problem-solving strategies. Mastering a math game can be hard work, but kids do it willingly because it is fun.

So what are you waiting for? Clear off a table, grab a deck of cards, and let's play some math!



Counting & Number Bonds: Math Games for Early Learners

Preschool to Second Grade: Young children can play with counting and number recognition, while older students explore place value, build number sense, and begin learning the basics of addition.



Addition & Subtraction: Math Games for Elementary Students

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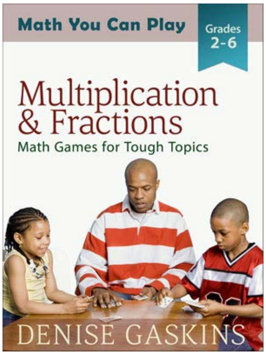
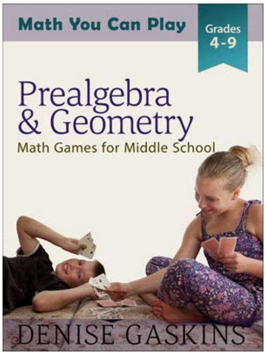
Multiplication & Fractions: Math Games for Tough Topics



Second to Sixth Grade: (planned for 2016) Students learn several math models that provide a sturdy foundation for understanding multiplication and fractions. The games feature times table facts and more advanced concepts such as division, fractions, decimals, and multistep mental math.



Prealgebra & Geometry: Math Games for Middle School

Fourth to Ninth Grade: (planned for 2017) Older students can handle more challenging games that develop logic and problem-solving skills. Here are playful ways to explore positive and negative integers, number properties, mixed operations, functions, and coordinate geometry.

Strategy & Reasoning: Games that Build Thinking Skills

For All Ages: (Planned for 2018.) Strategy games combine a relief from tedious textbook work with the adventure of creatively logical reasoning. Children who play strategy games learn to enjoy the challenge of thinking hard.